

Using Knowledge of How Students Think About Mathematics

Teachers informed about recent research on children's learning can modify their instruction to help students *construct* knowledge.

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A common metaphor is that the mind of the learner is like a tower of building blocks: the foundation must be built before higher blocks can be added. Accordingly (fortified by task analysis and behaviorism), most teaching in elementary school has rested on the assumption that students must learn lower-order facts and skills before they are able to master higher-order problem-solving and application skills.

Recent theory and research from cognitive psychology, however, suggest that knowledge is stored in the learner's head as a network of concepts or constructs: the mind of the learner is like a construction of tinker toys. Learning is the making of connections between new information and the learner's existing network of knowledge—the *construction* of knowledge by the learner—and instruction should facilitate these connections.

In a network theory of cognition and learning, the concepts of lower- and higher-order learning rarely make sense. For example, computational skills do not exist as lower-order prerequisites for higher-order mathematical problem solving, but rather are learned in relation to *and as part of* problem-solving (see, for example, Resnick 1985). Both "top-down" and "bottom-up" processes occur in learning mathematics.

If we are to foster the formation of these cognitive networks, new information must be related in a meaningful way to information that the learner already knows. Problem solving and meaningful learning are based on knowledge: people "continually try to understand and think about the new in terms of what they already know" (Glaser 1984, p. 100). Moreover, the specific content knowledge one already possesses plays a central role in one's thinking and learning.

New research on cognition and instruction can be used profitably to "formulate the principles that can guide interventions designed to help people learn" (Resnick 1985, p. 180). To help teachers apply this research in the classroom, we gave 1st grade teachers access to recent findings on children's mathematics learning (Carpenter

et al. 1988, Fennema et al. in press). The teachers then used this knowledge in their own ways to modify their teaching of addition and subtraction. We turn now to the research findings that we shared with the teachers.

Children's Informal Knowledge of Mathematics

When young children first begin to solve addition and subtraction word problems, they can only create a direct, concrete representation of the problem (Carpenter and Moser 1983, Riley et al. 1983). They typically use their fingers, physical objects, or counters to represent each quantity in the problem, and represent only the specific action or relationship described in the problem. For example, suppose we gave a kindergartner the following problem: "Melissa has three cookies. How many more does she need to have six cookies all together?" To solve this problem, the child might use counters (or spools, buttons, or rocks) to make a set of three objects, add more objects until she had a total of six, and then count the number of objects she had added to the original set.

Through interviewing children and studying how they talk about problems, researchers have found that children's problem-solving strategies become increasingly abstract as they are

able to engage in more abstract thinking. Children then begin to use more advanced counting strategies like "counting on" or "counting back." For example, to solve the cookie problem above, the child might use a "counting on" strategy if she recognized that it was not necessary to construct the first set of three objects. The child might simply count on from three to six, keeping track of the number of counts either on her fingers or with objects.

Children's Invented Mathematics Strategies

Eventually children memorize number facts that help them solve word problems. Children learn number facts over an extended period of time rather than all at once (Carpenter and Moser 1983). Also, they learn some number facts earlier than others, such as doubles like $5 + 5 = 10$. Most important, until they have memorized all the addition facts, many children naturally employ a small set of memorized facts to derive solutions for problems that have various number combinations. The following illustrates children's use of derived facts to solve a word problem:

Teacher: "Six frogs were sitting on lily pads. Eight more frogs joined them. How many frogs are there then?"

Rudy, Denise, Theo, and Sandra each answer "14" almost immediately.

Teacher: "How do you know there are 14?"

Rudy: "Because 6 and 6 is 12, and 2 more is 14."

Denise: "Eight and 8 is 16. But this is 8 and 6. That is 2 less, so it's 14."

Theo: "Well, I took 1 from the 8 and gave it to the 6. That made 7 and 7, and that's 14."

Sandra: "Eight and 2 is 10, and 4 more is 14."

Most children—not just the gifted—derive new facts from the facts they have already learned. Even without explicit instruction, children discover these strategies; and even with instruction that emphasizes symbol manipulation, children continue to rely directly on these strategies to represent problems and on counting techniques to solve them.

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Problem Type			
I. Join	A. Connie had 5 marbles. Jim gave her 8 more marbles. How many does Connie have all together? (result unknown)	B. Connie has 5 marbles. How many more marbles does she need to win to have 13 marbles all together? (change unknown)	C. Connie had some marbles. Jim gave her 5 more marbles. Now she has 13 marbles. How many marbles did Connie have to start with? (start unknown)
II. Separate	A. Connie had 13 marbles. She gave 5 marbles to Jim. How many marbles does she have left? (result unknown)	B. Connie had 13 marbles. She gave some to Jim. Now she has 5 marbles. How many did Connie give to Jim? (change unknown)	C. Connie had some marbles. She gave 5 to Jim. Now she has 8 marbles left. How many marbles did Connie have to start with? (start unknown)
III. Part-Part-Whole	A. Connie has 5 red marbles and 8 blue marbles. How many marbles does she have?	B. Connie has 13 marbles. Five are red and the rest are blue. How many blue marbles does Connie have?	
IV. Compare	A. Connie has 13 marbles. Jim has 5 marbles. How many more marbles does Connie have than Jim?	B. Jim has 5 marbles. Connie has 8 more than Jim. How many marbles does Connie have?	C. Connie has 13 marbles. She has 5 more marbles than Jim. How many marbles does Jim have?

Fig. 1. Classification of Word Problems

Children's Knowledge of Word Problems

In the eyes of children, not all addition or subtraction problems are alike. Children perceive important distinctions among different types of addition problems and among different types of subtraction problems. One way of classifying problems corresponds to the way children themselves think about problems. This classification scheme distinguishes problems that children solve in different ways and provides a framework to identify the relative difficulty of different problems (fig. 1).

In this model, there are four basic categories of addition and subtraction problems: Join, Separate, Part-Part-Whole, and Compare. Join and Separate problems involve action. In Join problems, elements are added to a given set. In Separate problems, elements are removed from a given set. Part-Part-Whole problems involve comparisons between two disjoint sets. Within each of these categories are several distinct types of problems that differ according to which quantity is the unknown: the initial quantity (start unknown), the second quantity (change unknown), or the final quantity (result unknown). Textbooks often include only Join Result Unknown and Separate Result Unknown problems, thus limiting children's ability to apply their mathematics skills to solve problems.

Principles for Classroom Instruction

From our reading of the research on 1st graders' learning and cognition in mathematics, we derived several principles for instruction:

- Rather than introducing story problems to children *after* they have mastered computational skills, teachers should use problem solving as the basis for teaching addition and subtraction.
- Teachers need to be familiar with a broad array of word problems and to know the processes that children usually use to solve different problems.
- Teachers can assess not only whether a child can solve a particular problem but also *how* the child solves the problem. Teachers need to analyze children's thinking by asking appropriate questions and by listening to children's responses.
- Teachers need to use the knowledge they derive from assessment and diagnosis of children to design appropriate instruction. Teachers should organize instruction so that children can easily and actively construct their own knowledge.
- Teachers should ensure that their mathematics instruction stresses relationships among concepts, skills, and problem solving.

We employed an instructional approach based on these principles called "Cognitively Guided Instruction" (CGI).

Teachers' Knowledge of Student Cognitions

We hypothesized that it might be important for teachers to know the mental processes, or cognitions, by which learners acquire specific subject matter knowledge in mathematics. Therefore, we investigated whether teachers knew the cognitions that 1st graders use to solve addition and subtraction problems. We wanted to answer these questions: (a) What do teachers know about distinctions that children naturally make between story problems in addition and subtraction? and (b) What do teachers know about the cognitions and strategies that children typically use to solve different addition and subtraction word problems?

In our study, we measured the knowledge of 40 1st grade teachers through questionnaires and interviews (Carpenter et al. in press). We found that most of the 1st grade teachers were able to identify the critical distinctions between addition and subtraction word problems (fig. 1), and the kinds of strategies that children use to solve such problems. However, the teachers had not organized their knowledge into a coherent network relating the different types of word problems to their difficulty and to children's cognitions for solving them.

An Experimental Study

To determine whether providing teachers access to recent research on chil-

dren's mathematics learning would cause them to modify their instruction, we conducted a controlled experiment. Forty 1st grade teachers agreed to participate (the same teachers whose knowledge we had surveyed). Half were assigned randomly to an experimental or CGI group, and half to a control group. Control teachers participated in a half-day workshop on problem solving. CGI teachers participated in a four-week summer workshop during which we presented recent findings on children's learning and cognition in addition and subtraction. We did *not* train the participants in either workshop in specific techniques for classroom instruction.

Following the workshops, these four assumptions elicited more agreement from CGI teachers than from control teachers:

1. Children construct their own mathematical knowledge.
2. Mathematics instruction should be organized to facilitate children's construction of knowledge.
3. Children's development of mathematical ideas should provide the basis for sequencing topics for instruction.
4. Mathematical skills should be taught in relation to understanding and problem solving.

We found that, as a result of the workshops, the knowledge base of CGI teachers was enhanced; and they subsequently changed their classroom instruction.

Effects on teachers' classroom instruction

The workshops took place in June 1986. During the 1986-87 school year, we observed the teachers and their students for four one-week periods, the first in October and the last in April. Both systematic observations and anecdotal evidence revealed differences between CGI and control teachers in their teaching of addition and subtraction. CGI teachers posed problems more often, and more often expected students to use multiple strategies rather than a single strategy to solve a problem. They often began the lesson by telling a story and then posed word problems based on the story. Control teachers' interactions

A CGI Teacher in Action

The following illustrates how one CGI teacher encouraged the use of multiple strategies to solve word problems that had practical meaning for students.

During the first few minutes of the day, Ms. White asked how many children wanted hot lunches that day. Eighteen children raised their hands. Six children were going to eat cold lunches. Ms. White asked, "How many children are going to eat lunch here today?"

By starting with 18 and counting on, several children got to the answer of 24. One child got out counters and counted out a set of 18 and another set of 6. He then counted all of them and said "24."

Ms. White then asked, "How many more children are eating hot lunch than are eating cold lunch?"

Several children counted back from 18 to 12. The child with the blocks matched 18 blocks with 6 blocks and counted the blocks left over.

Ms. White asked the children who volunteered to tell the rest of the class how they got the answer. Ms. White continued asking for different solutions until no one could think of a new way to solve the problem.

Children perceive important distinctions among different types of addition problems and among different types of subtraction problems.

more often concerned computations and number facts.

Effects on students' problem-solving and computational skills

In September and again in May, students in the CGI and control teachers' classes completed the Iowa Test of Basic Skills (ITBS) and an experimenter-constructed word-problem solving test. Regression analyses on students' post-test scores, adjusting for pretest achievement scores, showed that students in CGI teachers' classes had significantly greater ability to solve addition and subtraction word problems, particularly complex word problems. Further, a significant aptitude-treatment interaction (ATI) with prior

achievement showed that lower-achieving CGI classes especially benefited from the CGI approach, as shown by their achievement on a test of simple word problems. In addition, students in CGI classes were more confident of their ability to solve word problems than were students in control teachers' classes. Finally, although the observation data indicated that control teachers spent more time on addition and subtraction number facts than did CGI teachers, the students in CGI teachers' classes did as well as students in control teachers' classes on a computation test, and CGI students actually had better recall of number facts.

Extending Teachers' Knowledge Base

We draw three important conclusions from our study. First, a substantive knowledge base now exists regarding the psychology of children's classroom learning of mathematics. Second, even experienced teachers do not have this knowledge. Third, if teachers are given access to this knowledge, they can enhance their understanding of children's classroom learning in mathematics and improve their classroom instruction.

These experiences demonstrate that knowledge of the psychology of children's learning and cognition is useful to teachers. The challenge is to make

this research-based knowledge accessible to both practicing teachers and students in teacher education programs. If we do so, perhaps all teachers will be able to help their students actively *construct* knowledge. □

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