1 The Beauty in Numbers

We are accustomed to seeing numbers in charts and tables on the sports or business pages of a newspaper. We use numbers continuously in our everyday life experiences, either to represent a quantity or to designate something such as a street, address, or page. We use numbers without ever taking the time to observe some of their unusual properties. That is, we don't stop to smell the flowers as we walk through a garden, or as it is more commonly said: "take time to smell the roses." Inspecting some of these unusual number properties provides us with a much deeper appreciation for these symbols that we all too often take for granted. Students too often are taught mathematics as a dry and required course of instruction. As teachers, we have an obligation to make it interesting. To show some of the number oddities brings some new "life" to the subject. It will evoke a "gee whiz" response from students. That's what you ought to strive for. Make them curious about the subject. Motivate them to "dig" further.

There are basically two types of number properties, those that are "quirks" of the decimal system and those that are true in any number system. Naturally, the latter gives us better insight into mathematics, while the former merely points out the arbitrary nature of using a decimal system. One might ask why we use a decimal system (i.e., base 10) when today we find the foundation of computers relies on a binary system (i.e., base 2). The answer is clearly historical, and no doubt emanates from our number of fingers.

On the surface, the two types of peculiarities do not differ much in their appearance, just their justification. Since this book is intended for the average student's enjoyment (of course, presented appropriately), the justifications or explanations will be kept simple and adequately intelligible. By the same token, in some cases the explanation might lead the reader to further research into or inspection of the phenomenon. The moment you can bring students to the point where they question *why* the property exhibited occurred, they're hooked! That is the goal of this chapter, to make students want to marvel at the results and question them. Although the explanations may leave them with some questions, they will be well on their way to doing some individual explorations. That is when they really get to appreciate the mathematics involved. It is during these "private" investigations that genuine learning takes place. Encourage it!

Above all, they must take note of the beauty of the number relationships. Without further ado, let's go to the charming realm of numbers and number relationships.

1.1 Surprising Number Patterns I

There are times when the charm of mathematics lies in the surprising nature of its number system. There are not many words needed to demonstrate this charm. It is obvious from the patterns attained. Look, enjoy, and spread these amazing properties to your students. Let them appreciate the patterns and, if possible, try to look for an "explanation" for this. Most important is that the students can get an appreciation for the beauty in these number patterns.

$$\begin{array}{rcl}
1 \cdot 1 & = & 1 \\
11 \cdot 11 & = & 121 \\
111 \cdot 111 & = & 12{,}321 \\
1{,}111 \cdot 1{,}111 & = & 1{,}234{,}321 \\
11{,}111 \cdot 11{,}111 & = & 123{,}454{,}321 \\
111{,}111 \cdot 111{,}111 & = & 12{,}345{,}654{,}321 \\
1{,}111{,}111 \cdot 1{,}111{,}111 & = & 1{,}234{,}567{,}654{,}321 \\
11{,}111{,}111 \cdot 11{,}111{,}111 & = & 123{,}456{,}787{,}654{,}321 \\
111{,}111{,}111 \cdot 111{,}111{,}111 & = & 12{,}345{,}678{,}987{,}654{,}321
\end{array}$$

$$
\begin{aligned}
1 \cdot 8 + 1 &= 9 \\
12 \cdot 8 + 2 &= 98 \\
123 \cdot 8 + 3 &= 987 \\
1{,}234 \cdot 8 + 4 &= 9{,}876 \\
12{,}345 \cdot 8 + 5 &= 98{,}765 \\
123{,}456 \cdot 8 + 6 &= 987{,}654 \\
1{,}234{,}567 \cdot 8 + 7 &= 9{,}876{,}543 \\
12{,}345{,}678 \cdot 8 + 8 &= 98{,}765{,}432 \\
123{,}456{,}789 \cdot 8 + 9 &= 987{,}654{,}321
\end{aligned}
$$

Notice (below) how various products of 76,923 yield numbers in the same order but with a different starting point. Here the first digit of the product goes to the end of the number to form the next product. Otherwise, the order of the digits is intact.

$$
\begin{aligned}
76{,}923 \cdot 1 &= 076{,}923 \\
76{,}923 \cdot 10 &= 769{,}230 \\
76{,}923 \cdot 9 &= 692{,}307 \\
76{,}923 \cdot 12 &= 923{,}076 \\
76{,}923 \cdot 3 &= 230{,}769 \\
76{,}923 \cdot 4 &= 307{,}692
\end{aligned}
$$

Notice (below) how various products of 76,923 yield different numbers from those above, yet again, in the same order but with a different starting point. Again, the first digit of the product goes to the end of the number to form the next product. Otherwise, the order of the digits is intact.

$$
\begin{aligned}
76{,}923 \cdot 2 &= 153{,}846 \\
76{,}923 \cdot 7 &= 538{,}461 \\
76{,}923 \cdot 5 &= 384{,}615 \\
76{,}923 \cdot 11 &= 846{,}153 \\
76{,}923 \cdot 6 &= 461{,}538 \\
76{,}923 \cdot 8 &= 615{,}384
\end{aligned}
$$

Another peculiar number is 142,857. When it is multiplied by the numbers 2 through 8, the results are astonishing. Consider the following products and describe the peculiarity.

$142,857 \cdot 2 = 285,714$
$142,857 \cdot 3 = 428,571$
$142,857 \cdot 4 = 571,428$
$142,857 \cdot 5 = 714,285$
$142,857 \cdot 6 = 857,142$

You can see symmetries in the products but notice also that the same digits are used in the product as in the first factor. Furthermore, consider the order of the digits. With the exception of the starting point, they are in the same sequence.

Now look at the product, $142,857 \cdot 7 = 999,999$. Surprised?

It gets even stranger with the product, $142,857 \cdot 8 = 1,142,856$. If we remove the millions digit and add it to the units digit, the original number is formed.

It would be wise to allow the students to discover the patterns themselves. You can present a starting point or a hint at how they ought to start and then let them make the discoveries themselves. This will give them a sense of "ownership" in the discoveries. These are just a few numbers that yield strange products.

1.2 Surprising Number Patterns II

Here are some more charmers of mathematics that depend on the surprising nature of its number system. Again, not many words are needed to demonstrate the charm, for it is obvious at first sight. Just look, enjoy, and share these amazing properties with your students. Let them appreciate the patterns and, if possible, try to look for an "explanation" for this.

$$\begin{aligned}
12345679 \cdot 9 &= 111{,}111{,}111\\
12345679 \cdot 18 &= 222{,}222{,}222\\
12345679 \cdot 27 &= 333{,}333{,}333\\
12345679 \cdot 36 &= 444{,}444{,}444\\
12345679 \cdot 45 &= 555{,}555{,}555\\
12345679 \cdot 54 &= 666{,}666{,}666\\
12345679 \cdot 63 &= 777{,}777{,}777\\
12345679 \cdot 72 &= 888{,}888{,}888\\
12345679 \cdot 81 &= 999{,}999{,}999
\end{aligned}$$

In the following pattern chart, notice that the first and last digits of the products are the digits of the multiples of 9.

$$\begin{aligned}
987654321 \cdot 9 &= 08\ 888\ 888\ 889\\
987654321 \cdot 18 &= 17\ 777\ 777\ 778\\
987654321 \cdot 27 &= 26\ 666\ 666\ 667\\
987654321 \cdot 36 &= 35\ 555\ 555\ 556\\
987654321 \cdot 45 &= 44\ 444\ 444\ 445\\
987654321 \cdot 54 &= 53\ 333\ 333\ 334\\
987654321 \cdot 63 &= 62\ 222\ 222\ 223\\
987654321 \cdot 72 &= 71\ 111\ 111\ 112\\
987654321 \cdot 81 &= 80\ 000\ 000\ 001
\end{aligned}$$

It is normal for students to want to find extensions of this surprising pattern. They might experiment by adding digits to the first multiplicand or by multiplying by other multiples of 9. In any case, experimentation ought to be encouraged.

1.3 Surprising Number Patterns III

Here are some more charmers of mathematics that depend on the surprising nature of its number system. Again, not many words are needed to demonstrate the charm, for it is obvious at first sight. Just look, enjoy, and spread these amazing properties to your students. Let them appreciate the patterns and, if possible, try to look for an “explanation” for this. You might ask them why multiplying by 9 might give such unusual results. Once they see that 9 is one less than the base 10, they might get other ideas to develop multiplication patterns. A clue might be to have them consider multiplying by 11 (one greater than the base) to search for a pattern.

$$\begin{aligned}
0 \cdot 9 + 1 &= 1 \\
1 \cdot 9 + 2 &= 11 \\
12 \cdot 9 + 3 &= 111 \\
123 \cdot 9 + 4 &= 1{,}111 \\
1{,}234 \cdot 9 + 5 &= 11{,}111 \\
12{,}345 \cdot 9 + 6 &= 111{,}111 \\
123{,}456 \cdot 9 + 7 &= 1{,}111{,}111 \\
1{,}234{,}567 \cdot 9 + 8 &= 11{,}111{,}111 \\
12{,}345{,}678 \cdot 9 + 9 &= 111{,}111{,}111
\end{aligned}$$

A similar process yields another interesting pattern. Might this give your students more impetus to search further?

$$\begin{aligned}
0 \cdot 9 + 8 &= 8 \\
9 \cdot 9 + 7 &= 88 \\
98 \cdot 9 + 6 &= 888 \\
987 \cdot 9 + 5 &= 8{,}888 \\
9{,}876 \cdot 9 + 4 &= 88{,}888 \\
98{,}765 \cdot 9 + 3 &= 888{,}888 \\
987{,}654 \cdot 9 + 2 &= 8{,}888{,}888 \\
9{,}876{,}543 \cdot 9 + 1 &= 88{,}888{,}888 \\
98{,}765{,}432 \cdot 9 + 0 &= 888{,}888{,}888
\end{aligned}$$

Now the logical thing to inspect would be the pattern of these strange products.

$$
\begin{array}{rcr}
1 \cdot 8 & = & 8 \\
11 \cdot 88 & = & 968 \\
111 \cdot 888 & = & 98568 \\
1111 \cdot 8888 & = & 9874568 \\
11111 \cdot 88888 & = & 987634568 \\
111111 \cdot 888888 & = & 98765234568 \\
1111111 \cdot 8888888 & = & 9876541234568 \\
11111111 \cdot 88888888 & = & 987654301234568 \\
111111111 \cdot 888888888 & = & 98765431901234568 \\
1111111111 \cdot 8888888888 & = & 9876543217901234568
\end{array}
$$

How might you describe this pattern? Let students describe it in their own terms.

1.4 Surprising Number Patterns IV

Here are some more curiosities of mathematics that depend on the surprising nature of its number system. Again, not many words are needed to demonstrate the charm, for it is obvious at first sight. Yet in this case, you will notice that much is dependent on the number 1,001, which is the product of 7, 11, and 13. Furthermore, when your students multiply 1,001 by a three-digit number the result is nicely symmetric. For example, $987 \cdot 1{,}001 = 987{,}987$. Let them try a few of these on their own before proceeding.

Now let us reverse this relationship: Any six-digit number composed of two repeating sequences of three digits is divisible by 7, 11, and 13. For example,

$$\frac{643{,}643}{7} = 91{,}949$$

$$\frac{643,643}{11} = 58,513$$

$$\frac{643,643}{13} = 49,511$$

We can also draw another conclusion from this interesting number 1,001. That is, a number with six repeating digits is always divisible by 3, 7, 11, and 13. Here is one such example. Have your students verify our conjecture by trying others.

$$\frac{111,111}{3} = 37,037$$

$$\frac{111,111}{7} = 15,873$$

$$\frac{111,111}{11} = 10,101$$

$$\frac{111,111}{13} = 8,547$$

What other relationships can be found that play on the symmetric nature of 1,001?

1.5 Surprising Number Patterns V

Here are some more charmers of mathematics that depend on the surprising nature of its number system. Again, not many words are needed to demonstrate the charm, for it is obvious at first sight. These depend on the property described in Unit 1.4 and the unusual property of the number 9.

$$\begin{aligned}
999{,}999 \cdot 1 &= 0{,}999{,}999\\
999{,}999 \cdot 2 &= 1{,}999{,}998\\
999{,}999 \cdot 3 &= 2{,}999{,}997\\
999{,}999 \cdot 4 &= 3{,}999{,}996\\
999{,}999 \cdot 5 &= 4{,}999{,}995\\
999{,}999 \cdot 6 &= 5{,}999{,}994\\
999{,}999 \cdot 7 &= 6{,}999{,}993\\
999{,}999 \cdot 8 &= 7{,}999{,}992\\
999{,}999 \cdot 9 &= 8{,}999{,}991\\
999{,}999 \cdot 10 &= 9{,}999{,}990
\end{aligned}$$

Again, the number 9, which owes some of its unique properties to the fact that it is 1 less than the base 10, presents some nice peculiarities.

$$\begin{aligned}
9 \cdot 9 &= 81\\
99 \cdot 99 &= 9{,}801\\
999 \cdot 999 &= 998{,}001\\
9{,}999 \cdot 9{,}999 &= 99{,}980{,}001\\
99{,}999 \cdot 99{,}999 &= 9{,}999{,}800{,}001\\
999{,}999 \cdot 999{,}999 &= 999{,}998{,}000{,}001\\
9{,}999{,}999 \cdot 9{,}999{,}999 &= 99{,}999{,}980{,}000{,}001
\end{aligned}$$

While playing with the number 9, you might ask your students to find an eight-digit number in which no digit is repeated and which when multiplied by 9 yields a nine-digit number in which no digit is repeated. Here are a few correct choices:

$$\begin{aligned}
81{,}274{,}365 \cdot 9 &= 731{,}469{,}285\\
72{,}645{,}831 \cdot 9 &= 653{,}812{,}479\\
58{,}132{,}764 \cdot 9 &= 523{,}194{,}876\\
76{,}125{,}483 \cdot 9 &= 685{,}129{,}347
\end{aligned}$$

1.6 Surprising Number Patterns VI

Here is another nice pattern to further motivate your students to search on their own for other patterns in mathematics. Again, not many words are needed to demonstrate the beauty of this pattern, for it is obvious at first sight.

$$
\begin{aligned}
1 &= 1 &&= 1 \cdot 1 = 1^2\\
1+2+1 &= 2+2 &&= 2 \cdot 2 = 2^2\\
1+2+3+2+1 &= 3+3+3 &&= 3 \cdot 3 = 3^2\\
1+2+3+4+3+2+1 &= 4+4+4+4 &&= 4 \cdot 4 = 4^2\\
1+2+3+4+5+4+3+2+1 &= 5+5+5+5+5 &&= 5 \cdot 5 = 5^2\\
1+2+3+4+5+6+5+4+3+2+1 &= 6+6+6+6+6+6 &&= 6 \cdot 6 = 6^2\\
1+2+3+4+5+6+7+6+5+4+3+2+1 &= 7+7+7+7+7+7+7 &&= 7 \cdot 7 = 7^2\\
1+2+3+4+5+6+7+8+7+6+5+4+3+2+1 &= 8+8+8+8+8+8+8+8 &&= 8 \cdot 8 = 8^2\\
1+2+3+4+5+6+7+8+9+8+7+6+5+4+3+2+1 &= 9+9+9+9+9+9+9+9+9 &&= 9 \cdot 9 = 9^2
\end{aligned}
$$

1.7 Amazing Power Relationships

Our number system has many unusual features built into it. Discovering them can certainly be a rewarding experience. Most students need to be coaxed to look for these relationships. This is where the teacher comes in.

You might tell them about the famous mathematician Carl Friedrich Gauss (1777–1855), who had superior arithmetic abilities to see relationships and patterns that eluded even the brightest minds. He used these uncanny skills to conjecture and prove many very important mathematical theorems. Give your students a chance to "discover" relationships. Don't discourage the trivial discoveries, for they could lead to more profound results later on.

Show them the following relationship and ask them to describe what is going on here.

$$81 = (8 + 1)^2 = 9^2$$

Then ask them to see if there is another number for which this relationship might hold true. Don't wait too long before showing them the following.

$$4{,}913 = (4 + 9 + 1 + 3)^3 = 17^3$$

By now the students should realize that the sum of the digits of this number taken to a power equals the number. This is quite astonishing, as they will see if they try to find other examples.

The list below will provide you with lots of examples of these unusual numbers. Enjoy yourself!

Number		**(Sum of the digits)^n**	**Number**		**(Sum of the digits)^n**
81	=	9^2	34,012,224	=	18^6
			8,303,765,625	=	45^6
512	=	8^3	24,794,911,296	=	54^6
4,913	=	17^3	68,719,476,736	=	64^6
5,832	=	18^3			
17,576	=	26^3	612,220,032	=	18^7
19,683	=	27^3	10,460,353,203	=	27^7
			27,512,614,111	=	31^7
2,401	=	7^4	52,523,350,144	=	34^7
234,256	=	22^4	271,818,611,107	=	43^7
390,625	=	25^4	1,174,711,139,837	=	53^7
614,656	=	28^4	2,207,984,167,552	=	58^7
1,679,616	=	36^4	6,722,988,818,432	=	68^7
17,210,368	=	28^5	20,047,612,231,936	=	46^8
52,521,875	=	35^5	72,301,961,339,136	=	54^8
60,466,176	=	36^5	248,155,780,267,521	=	63^8
205,962,976	=	46^5			

$$20{,}864{,}448{,}472{,}975{,}628{,}947{,}226{,}005{,}981{,}267{,}194{,}447{,}042{,}584{,}001 = 207^{20}$$

1.8 Beautiful Number Relationships

Who said numbers can't form beautiful relationships! Showing your students some of these unique situations might give them the feeling that there is more to "numbers" than meets the eye. They should be encouraged not only to verify these relationships, but also to find others that can be considered "beautiful."

Notice the consecutive exponents.

$$135 = 1^1 + 3^2 + 5^3$$
$$175 = 1^1 + 7^2 + 5^3$$
$$518 = 5^1 + 1^2 + 8^3$$
$$598 = 5^1 + 9^2 + 8^3$$

Now, taken one place further, we get

$$1{,}306 = 1^1 + 3^2 + 0^3 + 6^4$$
$$1{,}676 = 1^1 + 6^2 + 7^3 + 6^4$$
$$2{,}427 = 2^1 + 4^2 + 2^3 + 7^4$$

The next ones are really amazing. Notice the relationship between the exponents and the numbers.*

$$3{,}435 = 3^3 + 4^4 + 3^3 + 5^5$$
$$438{,}579{,}088 = 4^4 + 3^3 + 8^8 + 5^5 + 7^7 + 9^9 + 0^0 + 8^8 + 8^8$$

Now it's up to the class to verify these and discover other beautiful relationships.

* In the second illustration, you will notice that, for convenience and for the sake of this unusual situation, we have considered 0^0 as though its value is 0, when, in fact, it is indeterminate.

1.9 Unusual Number Relationships

There are a number of unusual relationships between certain numbers (as represented in the decimal system). There is not much explanation for them. Just enjoy them and see if your students can find others.

We are going to present pairs of numbers where the product and the sum are reversals of each other. Present them one at a time to your students so that they can really appreciate them.

The two numbers		Their product	Their sum
9	9	81	18
3	24	72	27
2	47	94	49
2	497	994	499

Ask students if they can find another pair of numbers that exhibits this unusual property. (They may have difficulty with this.)

Here's another strange relationship*:

$$1 = 1!$$

$$2 = 2!$$

$$145 = 1! + 4! + 5!$$

$$40{,}585 = 4! + 0! + 5! + 8! + 5!$$

(Remember that $0! = 1$.)

That appears to be all of this sort that exists, so don't bother having students search for more.

* The exclamation mark is called a factorial and represents the product of consecutive integers from 1 to the number before the factorial symbol. That is, $n! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot \cdots \cdot (n-2)(n-1)n$.

1.10 Strange Equalities

There are times when the numbers speak more effectively than any explanation. Here is one such case. Just have your students look at these equalities and see if they can discover others of the same type.

$$1^1 + 6^1 + 8^1 = 15 = 2^1 + 4^1 + 9^1$$
$$1^2 + 6^2 + 8^2 = 101 = 2^2 + 4^2 + 9^2$$

$$1^1 + 5^1 + 8^1 + 12^1 = 26 = 2^1 + 3^1 + 10^1 + 11^1$$
$$1^2 + 5^2 + 8^2 + 12^2 = 234 = 2^2 + 3^2 + 10^2 + 11^2$$
$$1^3 + 5^3 + 8^3 + 12^3 = 2{,}366 = 2^3 + 3^3 + 10^3 + 11^3$$

$$1^1+5^1+8^1+12^1+18^1+19^1=63=2^1+3^1+9^1+13^1+16^1+20^1$$
$$1^2+5^2+8^2+12^2+18^2+19^2=919=2^2+3^2+9^2+13^2+16^2+20^2$$
$$1^3+5^3+8^3+12^3+18^3+19^3=15{,}057=2^3+3^3+9^3+13^3+16^3+20^3$$
$$1^4+5^4+8^4+12^4+18^4+19^4=260{,}755=2^4+3^4+9^4+13^4+16^4+20^4$$

Not much more one can say here. Your students will probably say Wow! If that is achieved, then you have met your goal.

1.11 The Amazing Number 1,089

This unit is about a number that has some truly exceptional properties. We begin by showing how it just happens to "pop up" when least expected. Begin by having your students, all working independently, select a three-digit number (where the units and hundreds digits are not the same) and follow these instructions:

1. Choose any three-digit number (where the units and hundreds digits are not the same).

We will do it with you here by arbitrarily selecting **825.**

2. Reverse the digits of this number you have selected.

We will continue here by reversing the digits of 825 to get **528**.

3. Subtract the two numbers (naturally, the larger minus the smaller).

Our calculated difference is $\mathbf{825 - 528 = 297}$.

4. Once again, reverse the digits of this difference.

Reversing the digits of 297 we get the number **792**.

5. Now, add your last two numbers.

We then add the last two numbers to get $\mathbf{297 + 792 = 1,089}$.

Their result should be the same* as ours even though their starting numbers were different from ours.

They will probably be astonished that regardless of which numbers they selected at the beginning, they got the same result as we did, 1,089.

How does this happen? Is this a "freak property" of this number? Did we do something devious in our calculations?

* If not, then you made a calculation error. Check it.

Unlike other numerical curiosities, which depended on a peculiarity of the decimal system, this illustration of a mathematical oddity depends on the operations. Before we explore (for the more motivated students) why this happens, we want you to be able to impress your students with a further property of this lovely number 1,089.

Let's look at the first nine multiples of 1,089:

$$1{,}089 \cdot 1 = 1{,}089$$
$$1{,}089 \cdot 2 = 2{,}178$$
$$1{,}089 \cdot 3 = 3{,}267$$
$$1{,}089 \cdot 4 = 4{,}356$$
$$1{,}089 \cdot 5 = 5{,}445$$
$$1{,}089 \cdot 6 = 6{,}534$$
$$1{,}089 \cdot 7 = 7{,}623$$
$$1{,}089 \cdot 8 = 8{,}712$$
$$1{,}089 \cdot 9 = 9{,}801$$

Do you notice a pattern among the products? Look at the first and ninth products. They are the reverses of one another. The second and the eighth are also reverses of one another. And so the pattern continues, until the fifth product is the reverse of itself, known as a palindromic number.*

Notice, in particular, that $1{,}089 \cdot 9 = 9{,}801$, which is the reversal of the original number. The same property holds for $10{,}989 \cdot 9 = 98{,}901$, and similarly, $109{,}989 \cdot 9 = 989{,}901$. Students will be quick to offer extensions to this. Your students should recognize by now that we altered the original 1,089 by inserting a 9 in the middle of the number, and extended that by inserting 99 in the middle of the 1,089. It would be nice to conclude from this that each of the following numbers have the same property: 1,099,989, 10,999,989, 109,999,989, 1,099,999,989, 10,999,999,989, and so on.

* We have more about palindromic numbers in Unit 1.16.

As a matter of fact, there is only one other number with four or fewer digits where a multiple of itself is equal to its reversal, and that is the number 2,178 (which just happens to be $2 \cdot 1{,}089$), since $2{,}178 \cdot 4 = 8{,}712$. Wouldn't it be nice if we could extend this as we did with the above example by inserting 9s into the middle of the number to generate other numbers that have the same property? Your students ought to be encouraged to try this independently and try to come to some conclusion. Yes, it is true that

$$\begin{aligned} 21{,}978 \cdot 4 &= 87{,}912 \\ 219{,}978 \cdot 4 &= 879{,}912 \\ 2{,}199{,}978 \cdot 4 &= 8{,}799{,}912 \\ 21{,}999{,}978 \cdot 4 &= 87{,}999{,}912 \\ 219{,}999{,}978 \cdot 4 &= 879{,}999{,}912 \\ 2{,}199{,}999{,}978 \cdot 4 &= 8{,}799{,}999{,}912 \\ &\vdots \end{aligned}$$

As if the number 1,089 didn't already have enough cute properties, here is another one that (sort of) extends from the 1,089: We will consider 1 and 89 and notice what happens when you take any number and get the sum of the squares of the digits and continue the same way. Each time, you will eventually reach 1 or 89. Take a look at some examples that follow.

$n = 30$:

$$\begin{aligned} &3^2 + 0^2 = 9 \to 9^2 = 81 \to 8^2 + 1^2 = 65 \to 6^2 + 5^2 = 61 \to \\ &6^2 + 1^2 = 37 \to 3^2 + 7^2 = 58 \to 5^2 + 8^2 = \mathbf{89} \to \\ &8^2 + 9^2 = 145 \to 1^2 + 4^2 + 5^2 = 42 \to 4^2 + 2^2 = 20 \to \\ &2^2 + 0^2 = 4 \to 4^2 = 16 \to 1^2 + 6^2 = 37 \to 3^2 + 7^2 = 58 \to \\ &5^2 + 8^2 = \mathbf{89} \to \mathbf{\ldots} \end{aligned}$$

Notice that, when we reached 89 for a second time, it is obvious that we are in a loop and that we will continuously get back to 89. For each of the following, we get into a loop that will continuously repeat.

$n = 31$: $3^2 + 1^2 = 10 \rightarrow 1^2 + 0^2 = \mathbf{1} \rightarrow 1^2 = \mathbf{1}$

$n = 32$: $3^2 + 2^2 = 13 \rightarrow 1^2 + 3^2 = 10 \rightarrow 1^2 + 0^2 = \mathbf{1} \rightarrow 1^2 = \mathbf{1}$

$n = 33$: $3^2 + 3^2 = 18 \rightarrow 1^2 + 8^2 = 65 \rightarrow 6^2 + 5^2 = 61 \rightarrow$
$6^2 + 1^2 = 37 \rightarrow 3^2 + 7^2 = 58 \rightarrow 5^2 + 8^2 = \mathbf{89} \rightarrow$
$8^2 + 9^2 = 145 \rightarrow 1^2 + 4^2 + 5^2 = 42 \rightarrow 4^2 + 2^2 = 20 \rightarrow$
$2^2 + 0^2 = 4 \rightarrow 4^2 = 16 \rightarrow 1^2 + 6^2 = 37 \rightarrow$
$3^2 + 7^2 = 58 \rightarrow 5^2 + 8^2 = \mathbf{89} \rightarrow \ldots$

$n = 80$: $8^2 + 0^2 = 64 \rightarrow 6^2 + 4^2 = 52 \rightarrow 5^2 + 2^2 = 29 \rightarrow$
$2^2 + 9^2 = 85 \rightarrow 8^2 + 5^2 = \mathbf{89} \rightarrow 8^2 + 9^2 = 145 \rightarrow$
$1^2 + 4^2 + 5^2 = 42 \rightarrow 4^2 + 2^2 = 20 \rightarrow 2^2 + 0^2 = 4 \rightarrow$
$4^2 = 16 \rightarrow 1^2 + 6^2 = 37 \rightarrow 3^2 + 7^2 = 58 \rightarrow$
$5^2 + 8^2 = \mathbf{89} \rightarrow \ldots$

$n = 81$: $8^2 + 1^2 = 65 \rightarrow 6^2 + 5^2 = 61 \rightarrow 6^2 + 1^2 = 37 \rightarrow$
$3^2 + 7^2 = 58 \rightarrow 5^2 + 8^2 = \mathbf{89} \rightarrow 8^2 + 9^2 = 145 \rightarrow$
$1^2 + 4^2 + 5^2 = 42 \rightarrow 4^2 + 2^2 = 20 \rightarrow 2^2 + 0^2 = 4 \rightarrow$
$4^2 = 16 \rightarrow 1^2 + 6^2 = 37 \rightarrow 3^2 + 7^2 = 58 \rightarrow$
$5^2 + 8^2 = \mathbf{89} \rightarrow \ldots$

$n = 82$: $8^2 + 2^2 = 68 \rightarrow 6^2 + 8^2 = 100 \rightarrow 1^2 + 0^2 + 0^2 = \mathbf{1} \rightarrow$
$1^2 = \mathbf{1}$

$n = 85$: $8^2 + 5^2 = \mathbf{89} \rightarrow 8^2 + 9^2 = 145 \rightarrow 1^2 + 4^2 + 5^2 = 42 \rightarrow$
$4^2 + 2^2 = 20 \rightarrow 2^2 + 0^2 = 4 \rightarrow 4^2 = 16 \rightarrow$
$1^2 + 6^2 = 37 \rightarrow 3^2 + 7^2 = 58 \rightarrow 5^2 + 8^2 = \mathbf{89} \rightarrow \ldots$

Now let's return to the original oddity of the number 1,089. We assumed that any number we chose would lead us to 1,089. Ask students how they can be sure. Well, they could try all possible three-digit numbers to see if it works. That would be tedious and not particularly elegant. An investigation of this oddity is within reach of a good elementary algebra student. So for the more ambitious students, who might be curious about this phenomenon, we will provide an algebraic explanation as to why it "works."

We shall represent the arbitrarily selected three-digit number, *htu*, as $100h + 10t + u$, where h represents the hundreds digit, t represents the tens digit, and u represents the units digit.

Let $h > u$, which would be the case in either the number you selected or the reverse of it. In the subtraction, $u - h < 0$; therefore, take 1 from the tens place (of the minuend), making the units place $10 + u$.

Since the tens digits of the two numbers to be subtracted are equal, and 1 was taken from the tens digit of the minuend, then the value of this digit is $10(t - 1)$. The hundreds digit of the minuend is $h - 1$, because 1 was taken away to enable subtraction in the tens place, making the value of the tens digit $10(t - 1) + 100 = 10(t + 9)$.

We can now do the first subtraction:

$$\boxed{\begin{array}{lll} 100(h-1) & +10(t+9) & +(u+10) \\ 100u & +10t & +h \\ \hline 100(h-u-1) & +10(9) & +u-h+10 \end{array}}$$

Reversing the digits of this difference gives us

$$\boxed{100(u - h + 10) + 10(9) + (h - u - 1)}$$

Now adding these last two expressions gives us

$$\boxed{100(9) + 10(18) + (10 - 1) = 1{,}089}$$

It is important to stress that algebra enables us to inspect the arithmetic process, regardless of the number.

Before we leave the number 1,089, we should point out to students that it has one other oddity, namely,

$$\boxed{33^2 = 1{,}089 = 65^2 - 56^2}$$

which is unique among two-digit numbers.

By this time your students must agree that there is a particular beauty in the number **1,089**.

1.12 The Irrepressible Number 1

This is not a trick. Yet mathematics does provide curiosities that appear to be magic. This is one that has baffled mathematicians for many years and still no one knows why it happens. Try it, you'll like it—or at least the students will!

Begin by asking your students to follow two rules as they work with any *arbitrarily* selected number.

If the number is *odd*, then multiply by 3 and add 1. If the number is *even*, then divide by 2.

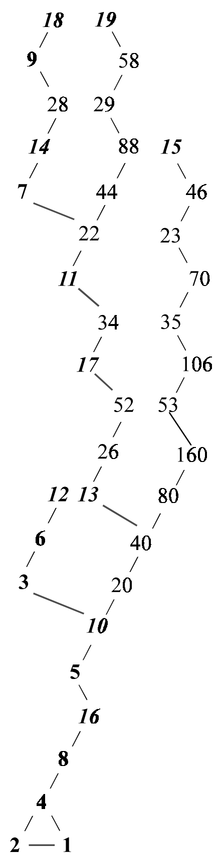
Regardless of the number they select, **they will always end up with 1**, after continued repetition of the process.

Let's try it for the *arbitrarily selected* number **12:**
12 is even; therefore, we divide by 2 to get 6.
6 is also even, so we again divide by 2 to get 3.
3 is odd; therefore, we multiply by 3 and add 1 to get $3 \cdot 3 + 1 = 10$.
10 is even, so we simply divide by 2 to get 5.
5 is odd, so we multiply by 3 and add 1 to get 16.
16 is even, so we divide by 2 to get 8.
8 is even, so we divide by 2 to get 4.
4 is even, so we divide by 2 to get 2.
2 is even, so we divide by 2 to get 1.

It is believed that, no matter which number we begin with (here we started with 12), we will eventually get to 1. This is truly remarkable! Try it for some other numbers to convince yourself that it really does work. Had we started with 17 as our arbitrarily selected number, we would have required 12 steps to reach 1. Starting with 43 will require 29 steps. You ought to have your students try this little scheme for any number they choose and see if they can get the number 1.

Does this really work for all numbers? This is a question that has concerned mathematicians since the 1930s, and to date no answer has been found, despite monetary rewards having been offered for a proof of this conjecture. Most recently (using computers) this problem, known in the literature as the "$3n + 1$" problem, has been shown to be true for the numbers up to $10^{18} - 1$.

For those who have been turned on by this curious number property, we offer you a schematic that shows the sequence of start numbers 1–20.



Notice that you will always end up with the final loop of 4–2–1. That is, when you reach 4 you will always get to the 1 and then were you to try to continue after having arrived at the 1, you will always get back to the 1, since, by applying the rule, $\boxed{3 \cdot 1 + 1 = 4}$ and you continue in the loop: 4–2–1.

We don't want to discourage inspection of this curiosity, but we want to warn you not to get frustrated if you cannot prove that it is true in all cases, for the best mathematical minds have not been able to do this for the better part of a century! Explain to your students that not all that we know or believe to be true in mathematics has been proved. There are still many "facts" that we must accept without proof, but we do so knowing that there may be a time when they will either be proved true for all cases, or someone will find a case for which a statement is not true, even after we have "accepted it."

1.13 Perfect Numbers

In mathematics, is there anything more perfect than something else? Most mathematics teachers constantly tell students that mathematics is perfect. Well, now we will introduce perfection in numbers—as it is defined by the mathematics community. According to tradition in number theory, we have an entity called a "perfect number." This is defined as a number equal to the sum of its proper factors (i.e., all the factors except the number itself). The smallest perfect number is 6, since $6 = 1 + 2 + 3$, which is the sum of all its proper factors.*

The next larger perfect number is 28, since again $28 = 1 + 2 + 4 + 7 + 14$. And the next one is $496 = 1 + 2 + 4 + 8 + 16 + 31 + 62 + 124 + 248$, which is the sum of all the proper factors of 496.

The first four perfect numbers were known to the Greeks. They are 6, 28, 496, and 8,128.

It was Euclid who came up with a theorem to generalize how to find a perfect number. He said that if $2^k - 1$ is a prime number, then $2^{k-1}(2^k - 1)$ is a perfect number. This is to say, whenever we find a value of k that gives us a prime for $2^k - 1$, then we can construct a perfect number.

* It is also the only number that is the sum and product of the same three numbers: $6 = 1 \cdot 2 \cdot 3 = 3!$ Also $6 = \sqrt{1^3 + 2^3 + 3^3}$. It is also interesting to notice that $\frac{1}{1} = \frac{1}{2} + \frac{1}{3} + \frac{1}{6}$. By the way, while on the number 6, it is nice to realize that both 6 and its square, 36, are triangular numbers (see Unit 1.17).

We do not have to use all values of k, since if k is a composite number, then $2^k - 1$ is also composite.*

Using Euclid's method for generating perfect numbers, we get the following table:

Values of k	Values of $2^{k-1}(2^k - 1)$ when $2^k - 1$ is a prime number
2	6
3	28
5	496
7	8,128
13	33,550,336
17	8,589,869,056
19	137,438,691,328

On observation, we notice some properties of perfect numbers. They all seem to end in either a 6 or a 28, and these are preceded by an odd digit. They also appear to be triangular numbers (see Unit 1.17), which are the sums of consecutive natural numbers (e.g., $496 = 1 + 2 + 3 + 4 + \cdots + 28 + 29 + 30 + 31$).

To take it a step further, every perfect number after 6 is the partial sum of the series: $1^3 + 3^3 + 5^3 + 7^3 + 9^3 + 11^3 + \cdots$. For example, $28 = 1^3 + 3^3$, and $496 = 1^3 + 3^3 + 5^3 + 7^3$. You might have your students try to find the partial sums for the next perfect numbers.

We do not know if there are any odd perfect numbers, but none has been found yet. Using today's computers, we have much greater facility at establishing more perfect numbers. Your students might try to find larger perfect numbers using Euclid's method.

* If $k = pq$, then $2^k - 1 = 2^{pq} - 1 = (2^p - 1)(2^{p(q-1)} + 2^{p(q-2)} + \cdots + 1)$. Therefore, $2^k - 1$ can only be prime when k is prime, but this does not guarantee that when k is prime, $2^k - 1$ will also be prime, as can be seen from the following values of k:

k	2	3	5	7	11	13
$2^k - 1$	3	7	31	127	2,047	8,191

where $2{,}047 = 23 \cdot 89$ is not a prime and so doesn't qualify.

1.14 Friendly Numbers

What could possibly make two numbers friendly? Your students' first reaction might be numbers that are friendly to them. Remind them that we are talking here about numbers that are "friendly" to each other. Well, mathematicians have decided that two numbers are considered friendly (or as often used in the more sophisticated literature, "amicable") if the sum of the proper divisors of one equals the second *and* the sum of the proper divisors of the second number equals the first number.

Sounds complicated? Have your students look at the smallest pair of friendly numbers: 220 and 284.

The proper divisors of 220 are 1, 2, 4, 5, 10, 11, 20, 22, 44, 55, and 110. Their sum is $1 + 2 + 4 + 5 + 10 + 11 + 20 + 22 + 44 + 55 + 110 = 284$.

The proper divisors of 284 are 1, 2, 4, 71, and 142, and their sum is $1 + 2 + 4 + 71 + 142 = 220$.
This shows the two numbers are friendly numbers.

The second pair of friendly numbers to be discovered (by Pierre de Fermat, 1601–1665) was 17,296 and 18,416:

$$17{,}296 = 2^4 \cdot 23 \cdot 47 \qquad \text{and} \qquad 18{,}416 = 2^4 \cdot 1{,}151$$

The sum of the proper factors of 17,296 is

$$\begin{aligned}1 + 2 + 4 + 8 + 16 + 23 + 46 + 47 + 92 + 94 + 184 + 188 + 368 \\ + 376 + 752 + 1{,}081 + 2{,}162 + 4{,}324 + 8{,}648 = 18{,}416\end{aligned}$$

The sum of the proper factors of 18,416 is

$$1 + 2 + 4 + 8 + 16 + 1{,}151 + 2{,}302 + 4{,}604 + 9{,}208 = 17{,}296$$

Here are a few more friendly pairs of numbers:

1,184 and 1,210
2,620 and 2,924
5,020 and 5,564
6,232 and 6,368
10,744 and 10,856
9,363,584 and 9,437,056
111,448,537,712 and 118,853,793,424

Your students might want to verify the above pairs' "friendliness"!

For the expert, the following is one method for finding friendly numbers. Let

$$a = 3 \cdot 2^n - 1$$
$$b = 3 \cdot 2^{n-1} - 1$$
$$c = 3^2 \cdot 2^{2n-1} - 1$$

where n is an integer greater than or equal to 2 and a, b, and c are all prime numbers. Then $2^n ab$ and $2^n c$ are friendly numbers.

(Notice that for $n \leq 200$, the values of $n = 2, 4$, and 7 give us a, b, and c to be prime.)

1.15 Another Friendly Pair of Numbers

We can always look for nice relationships between numbers. Some of them are truly mind-boggling! Take, for example, the pair of numbers: 6,205 and 3,869.

Guide your students to do the following to verify these fantastic results.

$$6{,}205 = 38^2 + 69^2 \quad \text{and} \quad 3{,}869 = 62^2 + 05^2$$

Notice the pattern and then follow with these numbers:

$$5{,}965 = 77^2 + 06^2 \quad \text{and} \quad 7{,}706 = 59^2 + 65^2$$

Beyond the enjoyment of seeing this wonderful pattern, there isn't much. However, the manner in which this is presented to the class can make all the difference!

1.16 Palindromic Numbers

It is sometimes nice to show your class some amusing mathematics that parallels amusing word games. Think of it not as time wasted, but rather as time spent to motivate youngsters to like mathematics more. A palindrome is a word, phrase, or sentence that reads the same in both directions. Here are a few amusing palindromes:

RADAR
REVIVER
ROTATOR
LEPERS REPEL
MADAM I'M ADAM
STEP NOT ON PETS
NO LEMONS, NO MELON
DENNIS AND EDNA SINNED
ABLE WAS I ERE I SAW ELBA
A MAN, A PLAN, A CANAL, PANAMA
SUMS ARE NOT SET AS A TEST ON ERASMUS

Palindromic numbers are those that read the same in both directions. This leads us to consider that dates can be a source for some symmetric inspection. For example, the year 2002 is a palindrome, as is 1991.* There were several dates in October 2001 that appeared as palindromes when written in American style: 10/1/01, 10/22/01, and others. In February, Europeans had the ultimate palindromic moment at 8:02 P.M. on February 20, 2002, since they would have written it as 20.02, 20-02-2002. It is a bit thought provoking to have students come up with other palindromic dates. You might ask them to list the palindromic dates closest to one another.

Looking further, the first four powers of 11 are palindromic numbers:

$$11^1 = 11$$
$$11^2 = 121$$
$$11^3 = 1{,}331$$
$$11^4 = 14{,}641$$

A palindromic number can be either a prime number or a composite number. For example, 151 is a prime palindrome and 171 is a composite palindrome. Yet with the exception of 11, a palindromic prime must have an odd number of digits. Have your students try to find some palindromic primes.

It is interesting to show students how a palindromic number can be generated from any given number. All they need to do is to continually add a number to its reversal (i.e., the number written in the reverse order of digits) until a palindrome is arrived at.

* Those of us who have lived through 1991 and 2002 will be the last generation who will have lived through two palindromic years for over the next 1,000 years (assuming the current level of longevity).

For example, a palindrome can be reached with a single addition such as with the starting number 23:

$$23 + 32 = 55, \quad \text{a palindrome}$$

Or it might take two steps, such as with the starting number 75:

$$75 + 57 = 132 \qquad 132 + 231 = 363, \quad \text{a palindrome}$$

Or it might take three steps, such as with the starting number 86:

$$86+68=154 \quad 154+451=605 \quad 605+506=1111, \quad \text{a palindrome}$$

The starting number 97 will require six steps to reach a palindrome, while the number 98 will require 24 steps. Be cautioned about using the starting number 196; this one will go far beyond your capabilities to reach a palindrome.

There are some lovely patterns when dealing with palindromic numbers. For example, numbers that yield palindromic cubes are palindromic themselves.

Students should be encouraged to find more properties of palindromic numbers*—they're fun to play with.

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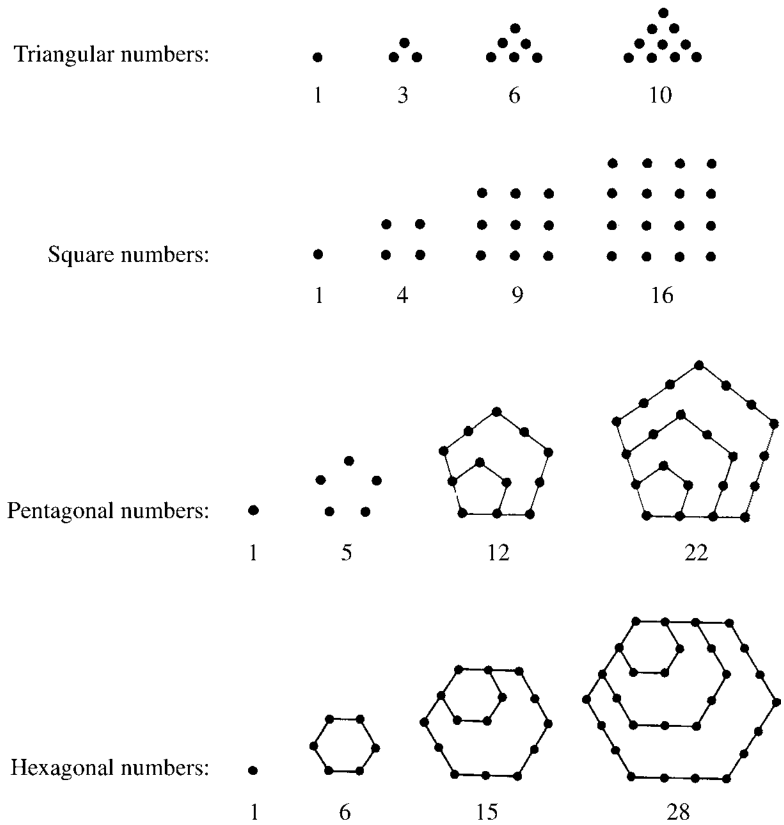
* One source for more information on palindromic numbers is A. S. Posamentier and J. Stepelman, *Teaching Secondary School Mathematics: Techniques and Enrichment Units,* 6th ed. (Upper Saddle River, NJ: Prentice Hall/Merrill, 2002), pp. 257–258.

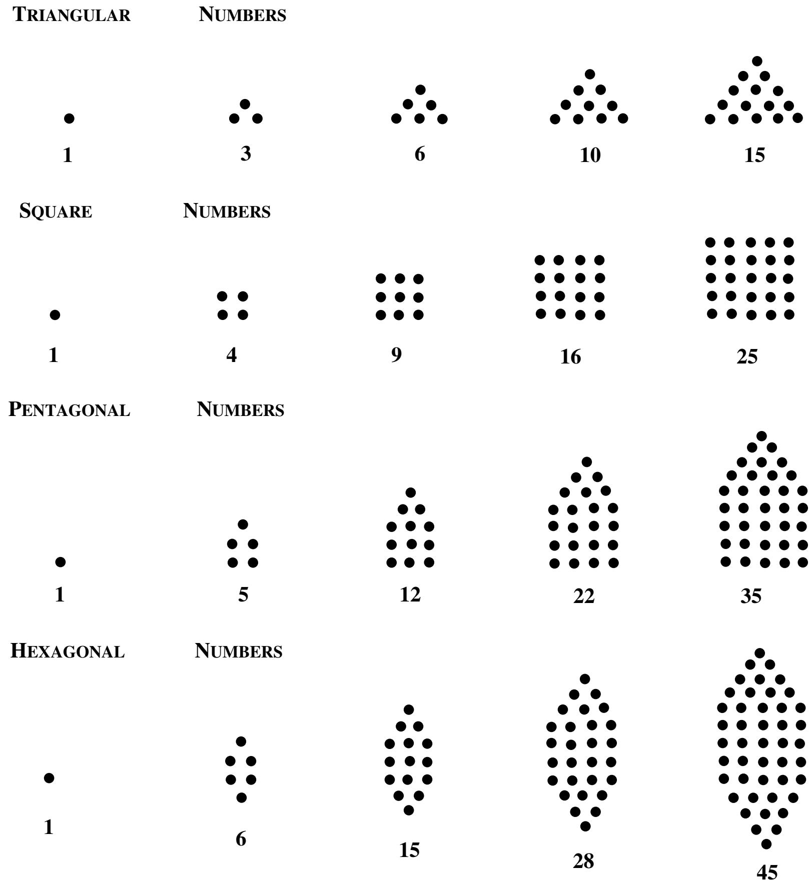
1.17 Fun with Figurate Numbers

How can numbers have a geometric shape? Well, although the numbers do not have a geometric shape, some can be represented by dots that can be put into a regular geometric shape. Let's take a look at some of these now.

Students should notice how the dots can be placed to form the shape of a regular polygon.



From the following arrangements of these figurate numbers, you ought to be able to discover some of their properties. It ought to be fun trying to relate these numbers to one another. For example, the nth square number is equal to the sum of the nth and the $(n-1)$th triangular numbers. Another example is that the nth pentagonal number is equal to the sum of the nth square number and the $(n-1)$th triangular number. There are lots of other such relationships to be found (or discovered!).



We can introduce students to *oblong numbers*, which look like $n(n+1)$, or rectangular arrays of dots such as

$$1 \cdot 2 = 2$$
$$2 \cdot 3 = 6$$
$$3 \cdot 4 = 12$$
$$4 \cdot 5 = 20$$
$$5 \cdot 6 = 30$$
$$\vdots$$

So here are some relationships involving oblong numbers; although examples are provided, your students should find additional examples to show these may be true. The more sophisticated can try to prove they are true.

An oblong number is the sum of consecutive even integers:

$$2 + 4 + 6 + 8 = 20$$

An oblong number is twice a triangular number:

$$15 \cdot 2 = 30$$

The sum of two consecutive squares and the square of the oblong between them is a square:

$$9 + 16 + 12^2 = 169 = 13^2$$

The sum of two consecutive oblong numbers and twice the square between them is a square:

$$12 + 20 + 2 \cdot 16 = 64 = 8^2$$

The sum of an oblong number and the next square is a triangular number:

$$20 + 25 = 45$$

The sum of a square number and the next oblong number is a triangular number:

$$25 + 30 = 55$$

The sum of a number and the square of that number is an oblong number:

$$9 + 81 = 90$$

Your students should now discover other connections between the various figurate numbers presented here.

1.18 The Fabulous Fibonacci Numbers

There aren't many themes in mathematics that permeate more branches of mathematics than the Fibonacci numbers. They come to us from one of the most important books in Western history. This book, *Liber abaci*, written in 1202 by Leonardo of Pisa, more popularly known as Fibonacci (1180–1250),* or son of Bonacci, is the first European publication using the Hindu–Arabic numerals that are the basis for our base 10 number system. This alone would qualify it as a landmark book. However, it also contains a "harmless" problem about the regeneration of rabbits. It is the solution of that problem that produces the Fibonacci numbers.

You might have your students try to set up a chart and solve the problem independently before progressing further. It may be stated as follows:

> **How many pairs of rabbits will be produced in a year, beginning with a single pair, if in every month each pair bears a new pair, which becomes productive from the second month on?**

It is from this problem that the famous *Fibonacci sequence* emerged. If we assume that a pair of baby (B) rabbits matures in one month to become

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* Fibonacci was not a clergyman, as might be expected of early scientists; rather, he was a merchant who traveled extensively throughout the Islamic world and took advantage of reading all he could of the Arabic mathematical writings. He was the first to introduce the Hindu–Arabic numerals to the Christian world in his *Liber abaci* (1202 and revised in 1228), which first circulated widely in manuscript form and was first published in 1857 as *Scritti di Leonardo Pisano* (Rome: B. Buoncompagni). The book is a collection of business mathematics, including linear and quadratic equations, square roots and cube roots, and other new topics, seen from the European viewpoint. He begins the book with the comment: "These are the nine figures of the Indians 9 8 7 6 5 4 3 2 1. With these nine figures, and with the symbol 0, which in Arabic is called *zephirum*, any number can be written, as will be demonstrated below". From here on, he introduces the decimal position system for the first time in Europe. (Note: The word "zephirum" evolved from the Arabic word *as-sifr*, which comes from the Sanskrit word, used in India as early as the fifth century, "sunya," referring to *empty*.)

offspring-producing adults (A), then we can set up the following chart:

Month	Pairs	Number of pairs of adults (A)	Number of pairs of babies (B)	Total pairs
January 1	A	1	0	1
February 1	A B	1	1	2
March 1	A B A	2	1	3
April 1	A B A A B	3	2	5
May 1	A B A A B A B A	5	3	8
June 1	A B A A B A B A A B A A B	8	5	13
July 1		13	8	21
August 1		21	13	34
September 1		34	21	55
October 1		55	34	89
November 1		89	55	144
December 1		144	89	233
January 1		233	144	377

The number of pairs of mature rabbits living each month determines the Fibonacci sequence (column 1): 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, ...

If we let f_n be the nth term of the Fibonacci sequence, then

$$\begin{aligned}
f_1 &= 1 \\
f_2 &= 1 \\
f_3 &= f_2 + f_1 = 1 + 1 = 2 \\
f_4 &= f_3 + f_2 = 2 + 1 = 3 \\
f_5 &= f_4 + f_3 = 3 + 2 = 5 \\
&\ \vdots \qquad\qquad \vdots \\
f_n &= f_{n-1} + f_{n-2} \quad \text{for } n \text{ an integer} \geq 3
\end{aligned}$$

That is, each term after the first two terms is the sum of the two preceding terms.

Your students may (rightly) ask at this point, What makes this sequence of numbers so spectacular? For one thing, there is a direct relationship between (believe it or not) it and the Golden Section! Consider successive quotients of the Fibonacci numbers:

$\frac{f_{n+1}}{f_n}$	$\frac{f_{n+1}}{f_n}$
$\frac{1}{1} = 1.000000000$	$\frac{55}{34} = 1.617647059$
$\frac{2}{1} = 2.000000000$	$\frac{89}{55} = 1.6182181618$
$\frac{3}{2} = 1.500000000$	$\frac{144}{89} = 1.617977528$
$\frac{5}{3} = 1.666666667$	$\frac{233}{144} = 1.618055556$
$\frac{8}{5} = 1.600000000$	$\frac{377}{233} = 1.618025751$
$\frac{13}{8} = 1.625000000$	$\frac{610}{377} = 1.618037135$
$\frac{21}{13} = 1.615384615$	$\frac{987}{610} = 1.618032787$
$\frac{34}{21} = 1.619047619$	

Furthermore, you can refer students to Unit 4.8 to notice that successive powers of ϕ^* present us with the Fibonacci numbers.

$$\phi^2 = \phi + 1$$
$$\phi^3 = 2\phi + 1$$
$$\phi^4 = 3\phi + 2$$
$$\phi^5 = 5\phi + 3$$
$$\phi^6 = 8\phi + 5$$
$$\phi^7 = 13\phi + 8$$

$^*\phi$ represents the Golden Ratio.

If, by now, the students didn't see the connection, highlight the coefficients and the constants. This is quite incredible; two completely (seemingly) unrelated things suddenly in close relationship to one another. That's what makes mathematics so wonderful!

1.19 Getting into an Endless Loop

This unit demonstrates an unusual phenomenon that arises out of the peculiarities of our decimal number system. There isn't much you can do with it, other than to marvel at the outcome. This is not something we can prove true for all cases; yet no numbers have been found for which it won't work. That, in itself, suffices to establish that it is apparently always true. You may wish to have your students use a calculator, unless you want them to have practice in subtraction. Here is how this procedure goes:

Begin by having them select a four-digit number (except one that has all digits the same).
Rearrange the digits of the number so that they form the largest number possible. Then rearrange the digits of the number so that they form the smallest number possible.
Subtract these two numbers (obviously, the smaller from the larger).
Take this difference and continue the process, over and over and over, until you notice something disturbing happening. Don't give up before something unusual happens.

Eventually, you will arrive at the number 6,174, perhaps after one subtraction or after several subtractions. When you do, you will find yourself in an endless loop.

When you have reached the loop, remind students that they began with a randomly selected number. Isn't this quite an astonishing result? Some students might be motivated to investigate this further. Others will just sit back in awe. Either way, they have been charmed again with the beauty of mathematics.

Here is an example of this activity.

We will (randomly) select the number 3,203.

The largest number formed with these digits is 3,320.
The smallest number formed with these digits is 0,233.
The difference is 3,087.

The largest number formed with these digits is 8,730.
The smallest number formed with these digits is 0,378.
The difference is 8,352.

The largest number formed with these digits is 8,532.
The smallest number formed with these digits is 2,358.
The difference is 6,174.

The largest number formed with these digits is 7,641.
The smallest number formed with these digits is 1,467.
The difference is 6,174.

And so the loop is formed, since you keep getting 6,174 if you continue.

1.20 A Power Loop

Can you imagine that a number is equal to the sum of the cubes of its digits? Take the time to explain exactly what this means. This should begin to "set them up" for this most unusual phenomenon. By the way, this is true for only five numbers. Below are these five most unusual numbers.

$$1 \rightarrow 1^3 = 1$$
$$153 \rightarrow 1^3 + 5^3 + 3^3 = 1 + 125 + 27 = 153$$
$$370 \rightarrow 3^3 + 7^3 + 0^3 = 27 + 343 + 0 = 370$$
$$371 \rightarrow 3^3 + 7^3 + 1^3 = 27 + 343 + 1 = 371$$
$$407 \rightarrow 4^3 + 0^3 + 7^3 = 64 + 0 + 343 = 407$$

Students should take a moment to appreciate these spectacular results and take note that these are the *only* such numbers for which this is true.

Taking sums of the powers of the digits of a number leads to interesting results. We can extend this procedure to get a lovely (and not to mention, surprising) technique you can use to have students familiarize themselves with powers of numbers and at the same time try to get to a startling conclusion.

Have them select any number and then find the sum of the cubes of the digits, just as we did previously. Of course, for any other number than those above, they will have reached a new number. They should then repeat this process with each succeeding sum until they get into a "loop." A loop can be easily recognized. When they reach a number that they reached earlier, then they are in a loop. This will become clearer with an example.

Let's begin with the number 352 and find the sum of the cubes of the digits.

The sum of the cubes of the digits of 352 is $3^3 + 5^3 + 2^3 = 27 + 125 + 8 = 160$. Now we use this sum, 160, and repeat the process:

The sum of the cubes of the digits of 160 is $1^3 + 6^3 + 0^3 = 1 + 216 + 0 = 217$. Again repeat the process with 217:

The sum of the cubes of the digits of 217 is $2^3 + 1^3 + 7^3 = 8 + 1 + 343 = 352$. Surprise! This is the same number (352) we started with.

You might think it would have been easier to begin by taking squares. You are in for a surprise. Let's try this with the number 123.

Beginning with 123, the sum of the squares of the digits is $1^2 + 2^2 + 3^2 = 1 + 4 + 9 = 14$.

1. Now using 14, the sum of the squares of the digits is $1^2 + 4^2 = 1 + 16 = 17$.
2. Now using 17, the sum of the squares of the digits is $1^2 + 7^2 = 1 + 49 = 50$.
3. Now using 50, the sum of the squares of the digits is $5^2 + 0^2 = 25$.
4. Now using 25, the sum of the squares of the digits is $2^2 + 5^2 = 4 + 25 = 29$.
5. Now using 29, the sum of the squares of the digits is $2^2 + 9^2 = 85$.

6. Now using 85, the sum of the squares of the digits is $8^2 + 5^2 = 64 + 25 = 89$.
7. Now using 89, the sum of the squares of the digits is $8^2 + 9^2 = 64 + 81 = 145$.
8. Now using 145, the sum of the squares of the digits is $1^2 + 4^2 + 5^2 = 1 + 16 + 25 = 42$.
9. Now using 42, the sum of the squares of the digits is $4^2 + 2^2 = 16 + 4 = 20$.
10. Now using 20, the sum of the squares of the digits is $2^2 + 0^2 = 4$.
11. Now using 4, the sum of the squares of the digits is $4^2 = 16$.
12. Now using 16, the sum of the squares of the digits is $1^2 + 6^2 = 1 + 36 = 37$.
13. Now using 37, the sum of the squares of the digits is $3^2 + 7^2 = 9 + 49 = 58$.
14. Now using 58, the sum of the squares of the digits is $5^2 + 8^2 = 25 + 64 = 89$.

Notice that the sum, 89, that we just got in step 14 is the same as in step 6, and so a repetition will now begin after step 14. This indicates that we would continue in a loop.

Students may want to experiment with the sums of the powers of the digits of any number and see what interesting results it may lead to. They should be encouraged to look for patterns of loops, and perhaps determine the extent of a loop based on the nature of the original number.

In any case, this intriguing unit can be fun just as it is presented here or it can be a source for further investigation by interested students.

1.21 A Factorial Loop

This charming little unit will show an unusual relationship for certain numbers. Before beginning, however, review with your class the definition of $n!$.

$$n! = 1 \cdot 2 \cdot 3 \cdot 4 \cdots (n-1) \cdot n$$

Now that they have an understanding of the factorial concept, have them find the sum of the factorials of the digits of 145.

$$1! + 4! + 5! = 1 + 24 + 120 = 145$$

Surprise! We're back to 145.

Only for certain numbers, will the sum of the factorials of the digits equal the number itself.

Have your students try this again with the number 40,585:

$$4! + 0! + 5! + 8! + 5! = 24 + 1 + 120 + 40{,}320 + 120 = 40{,}585$$

At this point, students will expect this to be true for just about any number. Well, just let them try another number. Chances are that it will not work.

Now have them try this scheme with the number 871. They will get

$$8! + 7! + 1! = 40{,}320 + 5{,}040 + 1 = 45{,}361$$

at which point they will feel that they have failed again.

Not so fast. Have them try this procedure again with 45,361. This will give them

$$4! + 5! + 3! + 6! + 1! = 24 + 120 + 6 + 720 + 1 = 871$$

Isn't this the very number we started with? Again, we formed a loop.

If they repeat this with the number 872, they will get

$$8! + 7! + 2! = 40{,}320 + 5{,}040 + 2 = 45{,}362$$

Then repeating the process will give them

$$4! + 5! + 3! + 6! + 2! = 24 + 120 + 6 + 720 + 2 = 872.$$

Again, we're in a loop.

Students are usually quick to form generalizations, so they might conclude that if the scheme of summing factorials of the digits of a number doesn't get you back to the original number then try it again and it ought to work. Of course, you can "stack the deck" by giving them the number 169 to try. Two cycles do not seem to present a loop. So have them proceed through one more cycle. And sure enough, the third cycle leads them back to the original number.

Starting number	Sum of the factorials
169	$1! + 6! + 9! = 363{,}601$
363,601	$3! + 6! + 3! + 6! + 0! + 1!$ $= 6 + 720 + 6 + 720 + 1 + 1 = 1{,}454$
1,454	$1! + 4! + 5! + 4!$ $= 1 + 24 + 120 + 24 = 169$

Be careful about having students draw conclusions. These factorial oddities are not so pervasive that you should tell students to find others. There are "within reach" three groups of such loops. We can organize them according to the number of times you have to repeat the process to reach the original number. We will call these repetitions "cycles."

Here is a summary of the way our numbers behave in this factorial loop.

1 cycle	1, 2, 145, 40,585
2 cycle	871, 45,361 and 872, 45,362
3 cycle	169, 363,601, 1,454

The factorial loops shown in this charming little number oddity can be fun, but students must be cautioned that there are no other such numbers less than 2,000,000 for which this works. So let them not waste their time. Just appreciate some little beauties!

1.22 The Irrationality of $\sqrt{2}$

When we say that $\sqrt{2}$ is irrational, what does that mean? Students should be encouraged to inspect the word "irrational" to determine its meaning in English.

Irrational means not rational.
Not rational means it cannot be expressed as a ratio of two integers.
Not expressible as a ratio means it cannot be expressed as a common fraction.
That is, there is *no* fraction $\frac{a}{b} = \sqrt{2}$ (where a and b are integers).

If we compute $\sqrt{2}$ with a calculator we will get

$$\begin{aligned}\sqrt{2} = 1.&41421356237309504880168872420969807856967187537694\\&80731766797379907324784621070388503875343276415 72\cdots\end{aligned}$$

Notice that there is no pattern among the digits, and there is no repetition of groups of digits. Does this mean that all rational fractions will have a period of digits*? Let's inspect a few common fractions.

$$\frac{1}{7} = 0.142857\underline{142857}142857\underline{142857}\cdots$$

which can be written as $0.\overline{142857}$ (a six-digit period).

Suppose we consider the fraction $\frac{1}{109}$:

$$\begin{aligned}\frac{1}{109} = \ &0.0091743119266055045871559633027522935779816513 76\\&14678899082568807339449541284403669724770642201834\\&8623\cdots\end{aligned}$$

Here we have calculated its value to more than 100 places and no period appears. Does this mean that the fraction is irrational? This would destroy our previous definition. We can try to calculate the value a bit more

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* A period of a sequence of digits is a group of repeating digits.

accurately, that is, say, to another 10 places further:

$$\begin{aligned}\frac{1}{109} = {} & 0.0091743119266055045871559633027522935779816513761 4\\ & 6788990825688073394495412844036697247706422018348623\\ & 853211\underline{0091}\end{aligned}$$

Suddenly it looks as though a pattern may be appearing; the 0091 also began the period.

We carry out our calculation further to 220 places and notice that, in fact, a 108-digit period emerges:

$$\begin{aligned}\frac{1}{109} = {} & 0.0091743119266055045871559633027522935779816513761\\ & 467889908256880733944954128440366972477064220183486\\ & 2385321\underline{1009174311926605504587155963302752293577981 6}\\ & \underline{51376146788990825688073394495412844036697247706422 0}\\ & \underline{18348623853211}009174\end{aligned}$$

If we carry out the calculation to 332 places, the pattern becomes clearer:

$$\begin{aligned}\frac{1}{109} = {} & 0.\mathbf{009174311926605504587155963302752293577981651376}\\ & \mathbf{146788990825688073394495412844036697247706422018 34}\\ & \mathbf{8623853211}0091743119266055045871559633027522935779\\ & 816513761467889908256880733944954128440366972477 06\\ & 422018348623853211\mathbf{0091743119266055045871559633027 5}\\ & \mathbf{229357798165137614678899082568807339449541284403 66}\\ & \mathbf{97247706422018348623853211}009174\end{aligned}$$

We might be able to conclude (albeit without proof) that a common fraction results in a decimal equivalent that has a repeating period of digits.

Some common ones we are already familiar with, such as

$$\frac{1}{3} = 0.33333333\overline{3}$$

$$\frac{1}{13} = 0.076923076923076923076923\overline{0769230}$$

To this point, we saw that a common fraction will result in a repeating decimal, sometimes with a very long period (e.g., $\frac{1}{109}$) and sometimes with a very short period (e.g., $\frac{1}{3}$). It would appear, from the rather flimsy evidence so far, that a fraction results in a repeating decimal and an irrational number does not. Yet this does not prove that an irrational number cannot be expressed as a fraction.

Here is a cute proof that $\sqrt{2}$ cannot be expressed as a common fraction and therefore, by definition is irrational.

> **Suppose $\frac{a}{b}$ is a fraction in lowest terms, which means that a and b do not have a common factor.**
> **Suppose $\frac{a}{b} = \sqrt{2}$. Then $\frac{a^2}{b^2} = 2$, or $a^2 = 2b^2$, which implies that a^2 and a are divisible by 2; written another way, $a = 2r$, where r is an integer.**
> **Then $4r^2 = 2b^2$, or $2r^2 = b^2$.**
> **So we have b^2 or b is divisible by 2.**
> **This contradicts the beginning assumption about the fact that a and b have no common factor, so $\sqrt{2}$ cannot be expressed as a common fraction.**

Understanding this proof may be a bit strenuous for some students, but a slow and careful step-by-step presentation should make it understandable for most algebra students.

1.23 Sums of Consecutive Integers

Ask your students: Which numbers can be expressed as the sum of consecutive integers? You may have your students try to establish a rule for this by trying to express the first batch of natural numbers as the sum of consecutive integers. We will provide some in the following list.

2 = not possible	$21 = 1 + 2 + 3 + 4 + 5 + 6$
$3 = 1 + 2$	$22 = 4 + 5 + 6 + 7$
4 = not possible	$23 = 11 + 12$
$5 = 2 + 3$	$24 = 7 + 8 + 9$
$6 = 1 + 2 + 3$	$25 = 12 + 13$
$7 = 3 + 4$	$26 = 5 + 6 + 7 + 8$
8 = not possible	$27 = 8 + 9 + 10$
$9 = 4 + 5$	$28 = 1 + 2 + 3 + 4 + 5 + 6 + 7$
$10 = 1 + 2 + 3 + 4$	$29 = 14 + 15$
$11 = 5 + 6$	$30 = 4 + 5 + 6 + 7 + 8$
$12 = 3 + 4 + 5$	$31 = 15 + 16$
$13 = 6 + 7$	32 = not possible
$14 = 2 + 3 + 4 + 5$	$33 = 10 + 11 + 12$
$15 = 4 + 5 + 6$	$34 = 7 + 8 + 9 + 10$
16 = not possible	$35 = 17 + 18$
$17 = 8 + 9$	$36 = 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8$
$18 = 5 + 6 + 7$	$37 = 18 + 19$
$19 = 9 + 10$	$38 = 8 + 9 + 10 + 11$
$20 = 2 + 3 + 4 + 5 + 6$	$39 = 19 + 20$
	$40 = 6 + 7 + 8 + 9 + 10$

These consecutive number sum representations are clearly not unique. For example, 30 can be expressed in other ways such as $9 + 10 + 11$ or $6 + 7 + 8 + 9$. An inspection of the table shows that those where a consecutive number sum was not possible were the powers of 2.

This is an interesting fact. It is not something that one would expect. By making a list of these consecutive number sums, students will begin to see patterns. Clearly, the triangular numbers are equal to the sum of the first n natural numbers. A multiple of 3, say $3n$, can always be represented by the sum: $(n - 1) + n + (n + 1)$. Students will discover other patterns.

That's part of the fun of it (not to mention its instructional value—seeing number patterns and relationships).

For the more ambitious students, we now will provide a proof of this (until-now) conjecture. First, we will establish when a number can be expressed as a sum of at least two consecutive positive integers.

Let us analyze what values can be taken by the sum of (two or more) consecutive positive integers from a to b ($b > a$)

$$S = a + (a+1) + (a+2) + \cdots + (b-1) + b = \left(\frac{a+b}{2}\right)(b-a+1)$$

by applying the formula for the sum of an arithmetic series.* Then, doubling both sides, we get:

$$2S = (a+b)(b-a+1)$$

Calling $(a+b) = x$ and $(b-a+1) = y$, we can note that x and y are both integers and that since their sum, $x + y = 2b + 1$, is odd, one of x, y is odd and the other is even. Note that $2S = xy$.

Case 1 S is a power of 2.

Let $S = 2^n$. We have $2(2^n) = xy$, or $2^{n+1} = xy$. The only way we can express 2^{n+1} as a product of an even and an odd number is if the odd number is 1. If $x = a + b = 1$, then a and b cannot be positive integers. If $y = b - a + 1 = 1$, then we have $a = b$, which also cannot occur. Therefore, S cannot be a power of 2.

Case 2 S is not a power of 2.

Let $S = m2^n$, where m is an odd number greater than 1. We have $2(m2^n) = xy$, or $m2^{n+1} = xy$. We will now find positive integers a and b such that $b > a$ and $S = a + (a+1) + \cdots + b$.

The two numbers 2^{n+1} and m are not equal, since one is odd and the other is even. Therefore, one is bigger than the other. Assign x to be

* $S = \frac{n}{2}(a + l)$, where n is the number of terms and a is the first term and l is the last term.

the bigger one and y to be the smaller one. This assignment gives us a solution for a and b, as $x + y = 2b + 1$, giving a positive integer value for b, and $x - y = 2a - 1$, giving a positive integer value for a. Also, $y = b - a + 1 > 1$, so $b > a$, as required. We have obtained a and b.

Therefore, for any S that is not a power of 2, we can find positive integers a and b, $b > a$, such that $S = a + (a + 1) + \cdots + b$.

In conclusion, a number can be expressed as a sum of (at least two) consecutive positive integers if and only if the number is not a power of 2.